Four Derivations of the Black-Scholes Formula

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In this note we derive in four separate ways the well-known result of Black and Scholes that under certain assumptions the time-t price $C(S_t, K, T)$ of a European call option with strike price K and maturity $\tau = T - t$ on a non-dividend stock with spot price S_t and a constant volatility σ when the rate of interest is a constant r can be expressed as

$$C(S_t, K, T) = S_t \Phi(d_1) - e^{-r\tau} K \Phi(d_2) \tag{1}$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

and $d_2 = d_1 - \sigma\sqrt{\tau}$, and where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} dt$ is the standard normal cdf. We show four ways in which Equation (1) can be derived.

1. By straightforward integration.
2. By applying the Feynman-Kac theorem.
3. By transforming the Black Scholes PDE into the heat equation, for which a solution is known. This is the original approach adopted by Black and Scholes [1].
4. Through the Capital Asset Pricing Model (CAPM).

Free code for the Black-Scholes model can be found at www.Volopta.com.

1 Black-Scholes Economy

There are two assets: a risky stock S and riskless bond B. These assets are driven by the SDEs

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r_t B_t dt \end{aligned} \tag{2}$$

The time zero value of the bond is $B_0 = 1$ and that of the stock is S_0. The model is valid under certain market assumptions that are described in John

Hull's book [3]. By Itō's Lemma the value V_t of a derivative written on the stock follows the diffusion

$$\begin{aligned} dV_t &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS)^2 \\ &= \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \\ &= \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\sigma S_t \frac{\partial V}{\partial S}\right) dW_t. \end{aligned} \tag{3}$$

2 The Lognormal Distribution

2.1 The Lognormal PDF and CDF

In this Note we make extensive use of the fact that if a random variable $Y \in \mathbb{R}$ follows the normal distribution with mean μ and variance σ^2, then $X = e^Y$ follows the lognormal distribution with mean

$$E[X] = e^{\mu + \frac{1}{2}\sigma^2} \tag{4}$$

and variance

$$Var[X] = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}. \tag{5}$$

The pdf for X is

$$dF_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) \tag{6}$$

and the cdf is

$$F_X(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right) \tag{7}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2} dt$ is the standard normal cdf.

2.2 The Lognormal Conditional Expected Value

The expected value of X conditional on $X > x$ is $L_X(K) = E[X|X > x]$. For the lognormal distribution this is, using Equation (6)

$$L_X(K) = \int_K^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2} dx.$$

Make the change of variable $y = \ln x$ so that $x = e^y$, $dx = e^y dy$ and the Jacobian is e^y. Hence we have

$$L_X(K) = \int_{\ln K}^\infty \frac{e^y}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y - \mu}{\sigma}\right)^2} dy. \tag{8}$$

Combining terms and completing the square, the exponent is

$$-\tfrac{1}{2\sigma^2}\left(y^2 - 2y\mu + \mu^2 - 2\sigma^2 y\right) = -\tfrac{1}{2\sigma^2}\left(y - \left(\mu + \sigma^2\right)\right)^2 + \mu + \tfrac{1}{2}\sigma^2.$$

Equation (8) becomes

$$L_X(K) = \exp\left(\mu + \tfrac{1}{2}\sigma^2\right)\frac{1}{\sigma}\int_{\ln K}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\frac{y-\left(\mu+\sigma^2\right)}{\sigma}\right)^2\right)dy. \tag{9}$$

Consider the random variable X with pdf $f_X(x)$ and cdf $F_X(x)$, and the scale-location transformation $Y = \sigma X + \mu$. It is easy to show that the Jacobian is $\frac{1}{\sigma}$, that the pdf for Y is $f_Y(y) = \frac{1}{\sigma} f_X\left(\frac{y-\mu}{\sigma}\right)$ and that the cdf is $F_Y(y) = F_X\left(\frac{y-\mu}{\sigma}\right)$. Hence, the integral in Equation (9) involves the scale-location transformation of the standard normal cdf. Using the fact that $\Phi(-x) = 1 - \Phi(x)$ this implies that

$$L_X(K) = \exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi\left(\frac{-\ln K + \mu + \sigma^2}{\sigma}\right). \tag{10}$$

See Hogg and Klugman [2].

3 Solving the SDEs

3.1 Stock Price

Apply Itō's Lemma to the function $\ln S_t$ where S_t is driven by the diffusion in Equation (2). Then $\ln S_t$ follows the SDE

$$d\ln S_t = \left(\mu - \tfrac{1}{2}\sigma^2\right)dt + \sigma dW_t. \tag{11}$$

Integrating from 0 to t, we have

$$\int_0^t d\ln S_u = \int_0^t \left(\mu - \tfrac{1}{2}\sigma^2\right)du + \sigma\int_0^t dW_u$$

so that

$$\ln S_t - \ln S_0 = \left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t$$

since $W_0 = 0$. Hence the solution to the SDE is

$$S_t = S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right). \tag{12}$$

Since W_t is distributed normal $N(0,t)$ with zero mean and variance t we have that $\ln S_t$ follows the normal distribution with mean $\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance $\sigma^2 t$. This implies by Equations (4) and (5) that S_t follows the lognormal distribution with mean $S_0 e^{\mu t}$ and variance $S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$. We can also integrate Equation (11) from t to T so that, analogous to Equation (12)

$$S_T = S_t \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)\tau + \sigma\left(W_T - W_t\right)\right) \tag{13}$$

and S_T follows the lognormal distribution with mean $S_t e^{\mu\tau}$ and variance given by $S_t^2 e^{2\mu\tau}\left(e^{\sigma^2\tau} - 1\right)$.

3.2 Bond Price

Apply Itō's Lemma to the function $\ln B_t$. Then $\ln B_t$ follows the SDE

$$d\ln B_t = r_t dt.$$

Integrating from 0 to t we have

$$d\ln B_t - d\ln B_0 = \int_0^t r_u du.$$

so the solution to the SDE is $B_t = \exp\left(\int_0^t r_u du\right)$ since $B_0 = 1$. When interest rates are constant then $r_t = r$ and $B_t = e^{rt}$. Integrating from t to T produces the solution $B_{t,T} = \exp\left(\int_t^T r_u du\right)$ or $B_{t,T} = e^{r\tau}$ when interest rates are constant.

3.3 Discounted Stock Price is a Martingale

We want to find a measure $\mathbb{Q}$ such that under $\mathbb{Q}$ the discounted stock price that uses B_t is a martingale. Write

$$dS_t = r_t S_t dt + \sigma S_t dW_t^{\mathbb{Q}} \tag{14}$$

where $W_t^{\mathbb{Q}} = W_t + \frac{\mu - r_t}{\sigma} t$. We have that under $\mathbb{Q}$, at time $t = 0$, the stock price S_t follows the lognormal distribution with mean $S_0 e^{r_t t}$ and variance $S_0^2 e^{2r_t t}\left(e^{\sigma^2 t} - 1\right)$, but that S_t is not a martingale. Using B_t as the numeraire, the discounted stock price is $\tilde{S}_t = \frac{S_t}{B_t}$ and $\tilde{S}_t$ will be a martingale. Apply Itō's Lemma to $\tilde{S}_t$, which follows the SDE

$$d\tilde{S}_t = \frac{\partial \tilde{S}}{\partial B} dB_t + \frac{\partial \tilde{S}}{\partial S} dS_t \tag{15}$$

since all terms involving the second-order derivatives are zero. Expand Equation (15) to obtain

$$\begin{aligned} d\tilde{S}_t &= -\frac{S_t}{B_t^2} dB_t + \frac{1}{B_t} dS_t \\ &= -\frac{S_t}{B_t^2}(r_t B_t dt) + \frac{1}{B_t}\left(r_t S_t dt + \sigma S_t dW_t^{\mathbb{Q}}\right) \\ &= \sigma \tilde{S}_t dW_t^{\mathbb{Q}}. \end{aligned} \tag{16}$$

The solution to the SDE (16) is

$$\tilde{S}_t = \tilde{S}_0 \exp\left(-\tfrac{1}{2}\sigma^2 t + \sigma W_t^{\mathbb{Q}}\right).$$

This implies that $\ln \tilde{S}_t$ follows the normal distribution with mean $\ln \tilde{S}_0 - \frac{\sigma^2}{2} t$ and variance $\sigma^2 t$. To show that $\tilde{S}_t$ is a martingale under $\mathbb{Q}$, consider the expectation under $\mathbb{Q}$ for $s < t$

$$\begin{aligned} E^{\mathbb{Q}}\left[\tilde{S}_t|F_s\right] &= \tilde{S}_0 \exp\left(-\tfrac{1}{2}\sigma^2 t\right) E^{\mathbb{Q}}\left[\exp\left(\sigma W_t^{\mathbb{Q}}\right)\Big|\mathcal{F}_s\right] \\ &= \tilde{S}_0 \exp\left(-\tfrac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{Q}}\right) E^{\mathbb{Q}}\left[\exp\left(\sigma\left(W_t^{\mathbb{Q}} - W_s^{\mathbb{Q}}\right)\right)\Big|\mathcal{F}_s\right] \end{aligned}$$

At time s we have that $W_t^{\mathbb{Q}} - W_s^{\mathbb{Q}}$ is distributed as $N(0, t-s)$ which is identical in distribution to $W_{t-s}^{\mathbb{Q}}$ at time zero. Hence we can write

$$E^{\mathbb{Q}}\left[\tilde{S}_t|\mathcal{F}_s\right] = \tilde{S}_0 \exp\left(-\tfrac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{Q}}\right) E^{\mathbb{Q}}\left[\exp\left(\sigma W_{t-s}^{\mathbb{Q}}\right)\Big|\mathcal{F}_0\right].$$

Now, the moment generating function (mgf) of a random variable X with normal distribution $N(\mu, \sigma^2)$ is $E\left[e^{\phi X}\right] = \exp\left(\mu\phi + \frac{1}{2}\phi^2\sigma^2\right)$. Under $\mathbb{Q}$ we have that $W_{t-s}^{\mathbb{Q}}$ is $\mathbb{Q}$-Brownian motion and distributed as $N(0, t-s)$. Hence the mgf of $W_{t-s}^{\mathbb{Q}}$ is $E^{\mathbb{Q}}\left[\exp\left(\sigma W_{t-s}^{\mathbb{Q}}\right)\right] = \exp\left(\frac{1}{2}\sigma^2 (t-s)\right)$ where σ takes the place of ϕ, and we can write

$$\begin{aligned} E^{\mathbb{Q}}\left[\tilde{S}_t|\mathcal{F}_s\right] &= \tilde{S}_0 \exp\left(-\tfrac{1}{2}\sigma^2 t + \sigma W_s^{\mathbb{Q}}\right) \exp\left(\tfrac{1}{2}\sigma^2 (t-s)\right) \\ &= \tilde{S}_0 \exp\left(-\tfrac{1}{2}\sigma^2 s + \sigma W_s^{\mathbb{Q}}\right) \\ &= \tilde{S}_s. \end{aligned}$$

We thus have that $E^{\mathbb{Q}}\left[\tilde{S}_t|\mathcal{F}_s\right] = \tilde{S}_s$, which shows that $\tilde{S}_t$ is a $\mathbb{Q}$-martingale. Pricing a European call option under Black-Scholes makes use of the fact that under $\mathbb{Q}$, at time t the terminal stock price at expiry, S_T, follows the normal distribution with mean $S_t e^{r\tau}$ and variance $S_t^2 e^{2r\tau}\left(e^{\sigma^2\tau} - 1\right)$ when the interest rate r_t is a constant value, r. Finally, note that under the original measure the process for $\tilde{S}_t$ is

$$d\tilde{S}_t = (\mu - r)\,\tilde{S}_t dt + \sigma \tilde{S}_t dW_t$$

which is obviously not a martingale.

3.4 Summary

We start with the processes for the stock price and bond price

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ dB_t &= r_t B_t dt. \end{aligned}$$

We apply Itō's Lemma to get the processes for $\ln S_t$ and $\ln B_t$

$$\begin{aligned} d\ln S_t &= \left(\mu - \tfrac{1}{2}\sigma^2\right) dt + \sigma dW_t \\ d\ln B_t &= r_t dt, \end{aligned}$$

which allows us to solve for S_t and B_t

$$\begin{aligned} S_t &= S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \\ B_t &= e^{\int_0^t r_s ds}. \end{aligned}$$

We apply a change of measure to obtain the stock price under the risk neutral measure $\mathbb{Q}$

$$\begin{aligned} dS_t &= rS_t + \sigma S_t dW_t^{\mathbb{Q}} \qquad \Rightarrow \\ S_t &= S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma W_t^{\mathbb{Q}}} \end{aligned}$$

Since S_t is not a martingale under $\mathbb{Q}$, we discount S_t by B_t to obtain $\tilde{S}_t = \frac{S_t}{B_t}$ and

$$\begin{aligned} d\tilde{S}_t &= \sigma \tilde{S}_t dW_t^{\mathbb{Q}} \qquad \Rightarrow \\ \tilde{S}_t &= \tilde{S}_0 e^{-\frac{1}{2}\sigma^2 t+\sigma W_t^{\mathbb{Q}}} \end{aligned}$$

so that $\tilde{S}_t$ *is* a martingale under $\mathbb{Q}$. The distributions of the processes described in this section are summarized in the following table

Stochastic Process	Lognormal distribution for $S_T \mid \mathcal{F}_t$		Process a martingale
	mean	variance	
$dS = \mu S dt + \sigma S dW$	$S_t e^{\mu\tau}$	$S_t^2 e^{2\mu\tau}\left(e^{\sigma^2\tau}-1\right)$	No
$dS = rSdt + \sigma S dW^{\mathbb{Q}}$	$S_t e^{r\tau}$	$S_t^2 e^{2r\tau}\left(e^{\sigma^2\tau}-1\right)$	No
$d\tilde{S} = \sigma\tilde{S}dW^{\mathbb{Q}}$ with $\tilde{S} = \frac{S}{B}$	$\tilde{S}_t$	$\tilde{S}_t^2\left(e^{\sigma^2\tau}-1\right)$	Yes
$d\tilde{S} = (\mu - r)\,\tilde{S}dt + \sigma\tilde{S}dW$	$S_t e^{(\mu-r)\tau}$	$S_t^2 e^{2(\mu-r)\tau}\left(e^{\sigma^2\tau}-1\right)$	No

This also implies that the logarithm of the stock price is normally distributed.

4 The Black-Scholes Call Price

In the following sections we show four ways in which the Black-Scholes call price can be obtained. Under a constant interest rate r the time-t price of a European call option on a non-dividend paying stock when its spot price is S_t and with strike K and time to maturity $\tau = T - t$ is

$$C(S_t, K, T) = e^{-r\tau} E^{\mathbb{Q}}\left[(S_T - K)^+ \middle| \mathcal{F}_t\right] \tag{17}$$

which can be evaluated to produce Equation (1), reproduced here for convenience

$$C(S_t, K, T) = S_t \Phi(d_1) - K e^{-r\tau}\Phi(d_2)$$

where

$$d_1 = \frac{\log \frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

and

$$\begin{aligned} d_2 &= d_1 - \sigma\sqrt{\tau} \\ &= \frac{\log \frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \end{aligned}$$

The first derivation is by straightforward integration of Equation (17); the second is by applying the Feynman-Kac theorem; the third is by transforming the Black-Scholes PDE into the heat equation and solving the heat equation; the fourth is by using the Capital Asset Pricing Model (CAPM).

5 Black-Scholes by Straightforward Integration

The European call price $C(S_t, K, T)$ is the discounted time-t expected value of $(S_T - K)^+$ under the EMM $\mathbb{Q}$ and when interest rates are constant. Hence from Equation (17) we have

$$\begin{aligned} C(S_t, K, T) &= e^{-r\tau} E^{\mathbb{Q}} \left[(S_T - K)^+ \middle| \mathcal{F}_t \right] \\ &= e^{-r\tau} \int_K^\infty (S_T - K) dF(S_T) \\ &= e^{-r\tau} \int_K^\infty S_T dF(S_T) - e^{-r\tau} K \int_K^\infty dF(S_T). \end{aligned} \tag{18}$$

To evaluate the two integrals, we make use of the result derived in Section (3.3) that under $\mathbb{Q}$ and at time t the terminal stock price S_T follows the lognormal distribution with mean $\ln S_t + \left(r - \frac{\sigma^2}{2} \right) \tau$ and variance $\sigma^2 \tau$, where $\tau = T - t$ is the time to maturity. The first integral in the last line of Equation (18) uses the conditional expectation of S_T given that $S_T > K$

$$\begin{aligned} \int_K^\infty S_T dF(S_T) &= E^{\mathbb{Q}} \left[S_T | S_T > K \right] \\ &= L_{S_T}(K). \end{aligned}$$

This conditional expectation is, from Equation (10)

$$\begin{aligned} L_{S_T}(K) &= \exp \left(\ln S_t + \left(r - \frac{\sigma^2}{2} \right) \tau + \frac{\sigma^2 \tau}{2} \right) \\ &\quad \times \Phi \left(\frac{-\ln K + \ln S_t + \left(r - \frac{\sigma^2}{2} \right) \tau + \sigma^2 \tau}{\sigma \sqrt{\tau}} \right) \\ &= S_t e^{r\tau} \Phi(d_1), \end{aligned}$$

so the first integral in the last line of Equation (18) is

$$S_t \Phi(d_1). \tag{19}$$

Using Equation (7), the second integral in the last line of (18) can be written

$$\begin{aligned} e^{-r\tau}K\int_K^\infty dF(S_T) &= e^{-r\tau}K\left[1-F(K)\right] \\ &= e^{-r\tau}K\left[1-\Phi\left(\frac{\ln K-\ln S_t-\left(r-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)\right] \\ &= e^{-r\tau}K\left[1-\Phi\left(-d_2\right)\right] \\ &= e^{-r\tau}K\Phi(d_2). \end{aligned} \tag{20}$$

Combining the terms in Equations (19) and (20) leads to the expression (1) for the European call price.

5.1 Change of Numeraire

The principle behind pricing by arbitrage is that if the market is complete we can find a portfolio that replicates the derivative at all times, and we can find an equivalent martingale measure (EMM) $\mathbb{N}$ such that the discounted stock price is a martingale. Moreover, the EMM $\mathbb{N}$ determines the unique numeraire N_t that discounts the stock price. The time-t value $V(S_t,t)$ of the derivative with payoff $V(S_T,T)$ at time T discounted by the numeraire N_t is

$$V(S_t,t)=N_t E^{\mathbb{N}}\left[\left.\frac{V(S_T,T)}{N_T}\right|\mathcal{F}_t\right]. \tag{21}$$

In the derivation of the previous section, the bond $B_t=e^{rt}$ serves as the numeraire, and since r is deterministic we can take $N_T=e^{rT}$ out of the expectation and with $V(S_T,T)=(S_T-K)^+$ we can write

$$V(S_t,t)=e^{-r(T-t)}E^{\mathbb{N}}\left[\left.(S_T-K)^+\right|\mathcal{F}_t\right]$$

which is Equation (17) for the call price.

5.1.1 Black Scholes Under a Different Numeraire

In this section we show that we can use the stock price S_t as the numeraire and recover the Black-Scholes call price. We start with the stock price process in Equation (14) under the measure $\mathbb{Q}$ and with a constant interest rate

$$dS_t=rS_t dt+\sigma S_t dW_t^{\mathbb{Q}}. \tag{22}$$

The relative bond price price is defined as $\tilde{B}=\frac{B}{S}$ and by Itō's Lemma follows the process

$$d\tilde{B}_t=\sigma^2\tilde{B}_t dt-\sigma\tilde{B}_t dW_t^{\mathbb{Q}}.$$

The measure $\mathbb{Q}$ turns $\tilde{S}=\frac{S}{B}$ into a martingale, but not $\tilde{B}$. The measure $\mathbb{P}$ that turns $\tilde{B}$ into a martingale is

$$W_t^{\mathbb{P}}=W_t^{\mathbb{Q}}-\sigma t \tag{23}$$

so that

$$d\tilde{B}_t = -\sigma \tilde{B}_t dW_t^{\mathbb{P}}$$

is a martingale under $\mathbb{P}$. The value of the European call is determined by using $N_t = S_t$ as the numeraire along with the payoff function $V(S_T, T) = (S_T - K)^+$ in the valuation Equation (21)

$$\begin{aligned} V(S_t, t) &= S_t E^{\mathbb{P}} \left[\left. \frac{(S_T - K)^+}{S_T} \right| \mathcal{F}_t \right] \\ &= S_t E^{\mathbb{P}} \left[(1 - K Z_T) | \mathcal{F}_t \right] \end{aligned} \tag{24}$$

where $Z_t = \frac{1}{S_t}$. To evaluate $V(S_t, t)$ we need the distribution for Z_T. The process for $Z = \frac{1}{S}$ is obtained using Itō's Lemma on S_t in Equation (22) and the change of measure in Equation (23)

$$\begin{aligned} dZ_t &= \left(-r + \sigma^2\right) Z_t dt - \sigma Z_t dW_t^{\mathbb{Q}} \\ &= -r Z_t dt - \sigma Z_t dW_t^{\mathbb{P}}. \end{aligned}$$

To find the solution for Z_t we define $Y_t = \ln Z_t$ and apply Itō's Lemma again, to produce

$$dY_t = -\left(r + \frac{\sigma^2}{2}\right) dt - \sigma dW_t^{\mathbb{P}}. \tag{25}$$

We integrate Equation (25) to produce the solution

$$Y_T - Y_t = -\left(r + \frac{\sigma^2}{2}\right)(T - t) - \sigma\left(W_T^{\mathbb{P}} - W_t^{\mathbb{P}}\right).$$

so that Z_T has the solution

$$Z_T = e^{\ln Z_t - \left(r + \frac{\sigma^2}{2}\right)(T-t) - \sigma\left(W_T^{\mathbb{P}} - W_t^{\mathbb{P}}\right)}. \tag{26}$$

Now, since $W_T^{\mathbb{P}} - W_t^{\mathbb{P}}$ is identical in distribution to $W_\tau^{\mathbb{P}}$, where $\tau = T - t$ is the time to maturity, and since $W_\tau^{\mathbb{P}}$ follows the normal distribution with zero mean and variance $\sigma^2 \tau$, the exponent in Equation (26)

$$\ln Z_t - \left(r + \frac{\sigma^2}{2}\right)(T - t) - \sigma\left(W_T^{\mathbb{P}} - W_t^{\mathbb{P}}\right),$$

follows the normal distribution with mean

$$u = \ln Z_t - \left(r + \frac{\sigma^2}{2}\right)\tau = -\ln S_t - \left(r + \frac{\sigma^2}{2}\right)\tau$$

and variance $v = \sigma^2 \tau$. This implies that Z_T follows the lognormal distribution with mean $e^{u+v/2}$ and variance $\left(e^v - 1\right) e^{2u+v}$. Note that $(1 - K Z_T)^+$ in the expectation of Equation (24) is non-zero when $Z_T < \frac{1}{K}$. Hence we can write this expectation as the two integrals

$$\begin{aligned} E^{\mathbb{P}}\left[(1 - K Z_T) | \mathcal{F}_t\right] &= \int_{-\infty}^{\frac{1}{K}} dF_{Z_T} - K \int_{-\infty}^{\frac{1}{K}} Z_T dF_{Z_T} \\ &= I_1 - I_2 \end{aligned} \tag{27}$$

where F_{Z_T} is the cdf of Z_T defined in Equation (7). The first integral in Equation (27) is

$$\begin{aligned} I_1 &= F_{Z_T}\left(\tfrac{1}{K}\right) = \Phi\left(\frac{\ln\frac{1}{K} - u}{v}\right) \\ &= \Phi\left(\frac{-\ln K + \ln S_t + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) \\ &= \Phi\left(d_1\right). \end{aligned} \tag{28}$$

Using the definition of $L_{Z_T}(x)$ in Equation (10), the second integral in Equation (27) is

$$\begin{aligned} I_2 &= K\left[\int_{-\infty}^{\infty} Z_T dF_{Z_T} - \int_{\frac{1}{K}}^{\infty} Z_T dF_{Z_T}\right] \\ &= K\left[E^{\mathbb{P}}\left[Z_T\right] - L_{Z_T}\left(\frac{1}{K}\right)\right] \\ &= K\left[e^{u+v/2} - e^{u+v/2}\Phi\left(\frac{-\ln\frac{1}{K} + u + v}{\sqrt{v}}\right)\right] \\ &= Ke^{u+v/2}\left[1 - \Phi\left(\frac{-\ln\frac{S_t}{K} - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)\right] \\ &= \frac{K}{S_t}e^{-r\tau}\Phi\left(d_2\right) \end{aligned} \tag{29}$$

since $1 - \Phi\left(-d_2\right) = \Phi\left(d_2\right)$. Substitute the expressions for I_1 and I_2 from Equations (28) and (29) into the valuation Equation (24)

$$\begin{aligned} V(S_t, t) &= S_t E^{\mathbb{P}}\left[\left(1 - KZ_T\right)\middle|\,\mathcal{F}_t\right] \\ &= S_t\left[I_1 - I_2\right] \\ &= S_t\Phi\left(d_1\right) - Ke^{-r\tau}\Phi\left(d_2\right) \end{aligned}$$

which is the Black-Scholes call price in Equation (1).

6 Black-Scholes From the Feynman-Kac Theorem

6.1 The Feynman-Kac Theorem

Suppose that x_t follows the process

$$dx_t = \mu(x_t, t)dt + \sigma\left(x_t, t\right)dW_t^{\mathbb{Q}} \tag{30}$$

and suppose the differentiable function $V = V(x_t, t)$ follows the partial differential equation given by

$$\frac{\partial V}{\partial t} + \mu(x_t, t)\frac{\partial V}{\partial x} + \tfrac{1}{2}\sigma(x_t, t)^2\frac{\partial^2 V}{\partial x^2} - r(t, x)V(x_t, t) = 0 \tag{31}$$

with boundary condition $V(x_T, T)$. The Feynman-Kac theorem stipulates that $V(x_t, t)$ has solution

$$V(x_t, t) = E^{\mathbb{Q}}\left[e^{-\int_t^T r(X_u, u)du} V(X_T, T) \middle| \mathcal{F}_t \right]. \tag{32}$$

In Equation (32) the time-t expectation is with respect to the same measure $\mathbb{Q}$ under which the stochastic portion of Equation (30) is Brownian motion. See the Note on www.FRouah.com for illustrations of the Feynman-Kac theorem.

6.2 The Theorem Applied to Black-Scholes

To apply the Feynman-Kac theorem to the Black-Scholes call price, note that the value $V_t = V(S_t, t)$ of a European call option written at time t with strike price K when interest rates are a constant r follows the Black-Scholes PDE

$$\frac{\partial V}{\partial t} + rS_t\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2} - rV_t = 0 \tag{33}$$

with boundary condition $V(S_T, T) = (S_T - K)^+$. The Note on www.FRouah.com explains how the PDE (33) is derived. This PDE is the PDE in Equation (31) with $x_t = S_t$, $\mu(x_t, t) = rS_t$, and $\sigma(x_t, t) = \sigma S_t$. Hence the Feynman-Kac theorem applies and the value of the European call is

$$\begin{aligned} V(S_t, t) &= E^{\mathbb{Q}}\left[e^{-\int_t^T r(X_u, u)du} V(S_T, T) \middle| \mathcal{F}_t \right] \\ &= e^{-r\tau} E^{\mathbb{Q}}\left[(S_T - K)^+ \middle| \mathcal{F}_t \right] \end{aligned} \tag{34}$$

which is exactly Equation (18). Hence, we can evaluate the expectation in (34) by straightforward integration exactly in the same way as in Section 5 and obtain the call price in Equation (1).

7 Black-Scholes From the Heat Equation

In this section we follow the derivation explained in Wilmott et al. [4]. We first present a definition of the Dirac delta function, and of the heat equation. We then transform the Black-Scholes PDE into the heat equation, apply the solution through integration, and convert back to the original (untransformed) parameters. This will produce the Black-Scholes call price.

7.1 Dirac Delta Function

The Dirac delta function $\delta(x)$ is defined as

$$\delta(x) = \begin{cases} 0 \text{ if } x \neq 0 \\ \infty \text{ if } x = 0 \end{cases} \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(\xi) d\xi = 1.$$

For an integrable function $f(x)$ we have that

$$f(0) = \int_{-\infty}^{\infty} f(\xi)\delta(\xi)d\xi$$

and

$$f(x) = \int_{-\infty}^{\infty} f(x-\xi)\delta(\xi)d\xi. \tag{35}$$

7.2 The Heat Equation

The heat equation is the PDE for $u = u(x, \tau)$ over the domain $\{x \in R, \tau > 0\}$ given by

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

The heat equation has the fundamental solution

$$u(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} \exp\left(\frac{-x^2}{4\tau}\right) \tag{36}$$

which is the normal pdf with mean 0 and variance 2τ. The initial value of the heat equation is $u(x, 0) = u_0(x)$ which can be written in terms of the Dirac delta function δ as the limit

$$u_0(x) = \lim_{\tau \to 0} u(x, \tau) = \delta(x).$$

Using the property (35) of the Dirac delta function, we can write the initial value as

$$u_0(x) = \int_{-\infty}^{\infty} \delta(x-\xi)u_0(\xi)d\xi \tag{37}$$

We can also apply the property (35) to the fundamental solution in Equation (36) and express the solution as

$$\begin{aligned} u(x, \tau) &= \int_{-\infty}^{\infty} u(x-\xi)\delta(\xi)d\xi \qquad (38) \\ &= \int_{-\infty}^{\infty} u(x-\xi)u_0(\xi)d\xi \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\tau}} e^{-(x-\xi)^2/4\tau} u_0(\xi)d\xi, \end{aligned}$$

with initial value

$$u(x, 0) = \int_{-\infty}^{\infty} \delta(x-\xi)u_0(\xi)d\xi = u_0(x),$$

as before.

7.3 The Black-Scholes PDE as the Heat Equation

Through a series of transformations we convert the Black-Scholes PDE in Equation (33) into the heat equation. The first set of transformations convert the spot price to log-moneyness and the time to one-half the total variance. This will get rid of the S and S^2 terms in the Black-Scholes PDE. The first transformations are

$$\begin{aligned} x &= \ln\frac{S}{K} \quad \text{so that} \quad S = Ke^x \qquad (39)\\ \tau &= \tfrac{\sigma^2}{2}(T-t) \quad \text{so that } t = T - \frac{2\tau}{\sigma^2} \\ U(x,\tau) &= \tfrac{1}{K}V(S,t) = \tfrac{1}{K}V\left(Ke^x, T-2\tau/\sigma^2\right). \end{aligned}$$

Apply the chain rule to the partial derivatives in the Black-Scholes PDE. We have

$$\begin{aligned} \frac{\partial V}{\partial t} &= K\frac{\partial U}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{-K\sigma^2}{2}\frac{\partial U}{\partial \tau}, \\ \frac{\partial V}{\partial S} &= K\frac{\partial U}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial U}{\partial x} = e^{-x}\frac{\partial U}{\partial x}, \\ \frac{\partial^2 V}{\partial S^2} &= -\frac{K}{S^2}\frac{\partial U}{\partial x} + \frac{K}{S}\frac{\partial}{\partial S}\left(\frac{\partial U}{\partial x}\right) \\ &= -\frac{K}{S^2}\frac{\partial U}{\partial x} + \frac{K}{S}\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)\frac{\partial x}{\partial S} \\ &= -\frac{K}{S^2}\frac{\partial U}{\partial x} + \frac{K}{S^2}\frac{\partial^2 U}{\partial x^2} \\ &= \frac{e^{-2x}}{K}\left(\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x}\right). \end{aligned}$$

Substitute for the partials in the Black-Scholes PDE (33) to obtain

$$\frac{-K\sigma^2}{2}\frac{\partial U}{\partial \tau} + rKe^x e^{-x}\frac{\partial U}{\partial x} + \frac{1}{2}\sigma^2 K^2 e^{2x}\frac{e^{-2x}}{K}\left(\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x}\right) - rU = 0$$

which simplifies to

$$-\frac{\partial U}{\partial \tau} + (k-1)\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2} - kU = 0 \qquad (40)$$

where $k = \frac{2r}{\sigma^2}$. The coefficients of this PDE does not involve x or τ. The boundary condition for V is $V(S_T,T) = (S_T - K)^+$. From Equation (39), when $t = T$ and $S_t = S_T$ we have that $x = \ln\frac{S_T}{K}$ which we write as x_T, and that $\tau = 0$. Hence the boundary condition for U is

$$U_0(x_T) = U(x_T,0) = \frac{1}{K}V(S_T-K)^+ = \frac{1}{K}(Ke^{x_T} - K)^+ = (e^{x_T} - 1)^+.$$

We make the additional transformation

$$W\,(x,\tau) = e^{\alpha x+\beta^2\tau}U(x,\tau) \tag{41}$$

where $\alpha = \frac{1}{2}(k-1)$ and $\beta = \frac{1}{2}(k+1)$. This will convert Equation (40) into the heat equation. The partial derivatives of U in terms of W are

$$\begin{aligned}
\frac{\partial U}{\partial \tau} &= e^{-\alpha x-\beta^2\tau}\left(\frac{\partial W}{\partial \tau} - W(x,\tau)\beta^2\right) \\
\frac{\partial U}{\partial x} &= e^{-\alpha x-\beta^2\tau}\left(\frac{\partial W}{\partial x} - \alpha W(x,\tau)\right) \\
\frac{\partial^2 U}{\partial x^2} &= e^{-\alpha x-\beta^2\tau}\left(\alpha^2 W(x,\tau) - 2\alpha\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2}\right).
\end{aligned}$$

Substitute these derivatives into Equation (40) to obtain

$$\beta^2 W(x,\tau) - \frac{\partial W}{\partial \tau} + (k-1)\left[-\alpha W(x,\tau) + \frac{\partial W}{\partial x}\right]$$
$$+\alpha W(x,\tau) - 2\alpha\frac{\partial W}{\partial x} + \frac{\partial^2 W}{\partial x^2} - kW(x,\tau) = 0$$

which simplifies to the heat equation

$$\frac{\partial W}{\partial \tau} = \frac{\partial W^2}{\partial x^2}. \tag{42}$$

From Equation (41) the boundary condition for $W(x,\tau)$ is

$$\begin{aligned}
W_0(x_T) &= W(x_T,0) = e^{\alpha x_T}U(x_T,0) \\
&= \left(e^{(\alpha+1)x_T} - e^{\alpha x_T}\right)^+ = \left(e^{\beta x_T} - e^{\alpha x_T}\right)^+.
\end{aligned} \tag{43}$$

since $\beta = \alpha + 1$. The transformation from V to W is therefore

$$V(S,t) = \frac{1}{K}e^{-\alpha x-\beta^2\tau}W(x,\tau). \tag{44}$$

7.4 Obtaining the Black-Scholes Call Price

Since $W(x,\tau)$ follows the heat equation, it has the solution given by Equation (38), with boundary condition given by (43). Hence the solution is

$$\begin{aligned}
W\,(x,\tau) &= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4\tau}W_0(\xi)d\xi \\
&= \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-(\xi-x)^2/4\tau}\left(e^{\beta\xi} - e^{\alpha\xi}\right)^+ d\xi.
\end{aligned}$$

Make the change of variable $z = \frac{\xi - x}{\sqrt{2\tau}}$ so that $\xi = \sqrt{2\tau}z + x$ and $d\xi = \sqrt{2\tau}dz$.

$$\begin{aligned} W(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \\ &\quad \times \exp\left(\beta\left[\sqrt{2\tau}z + x\right] - \alpha\left[\sqrt{2\tau}z + x\right]\right)^{+} dz \end{aligned} \tag{45}$$

Note that the integral is non-zero only when the second exponent is greater than zero, that is, when $\beta\left[\sqrt{2\tau}z + x\right] > \alpha\left[\sqrt{2\tau}z + x\right]$ which is identical to $z > \frac{-x}{\sqrt{2\pi}}$. We can now break up the integral into two pieces

$$\begin{aligned} W(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \exp\left(\beta\left[\sqrt{2\tau}z + x\right]\right) dz \\ &\quad - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \exp\left(\alpha\left[\sqrt{2\tau}z + x\right]\right) dz \\ &= I_1 - I_2. \end{aligned}$$

Complete the square in the first integral I_1. The exponent in the integrand is

$$-\frac{1}{2}z^2 + \beta\sqrt{2\tau}z + \beta x = -\frac{1}{2}\left(z - \beta\sqrt{2\tau}\right)^2 + \beta x + \beta^2\tau.$$

The first integral becomes

$$I_1 = e^{\beta x + \beta^2\tau} \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\left(z - \beta\sqrt{2\tau}\right)^2} dz.$$

Make the transformation $y = z - \beta\sqrt{2\tau}$ so that the integral becomes

$$\begin{aligned} I_1 &= e^{\beta x + \beta^2\tau} \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau} - \beta\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}y^2} dy \\ &= e^{\beta x + \beta^2\tau} \left(1 - \Phi\left(-\frac{x}{\sqrt{2\tau}} - \beta\sqrt{2\tau}\right)\right) \\ &= e^{\beta x + \beta^2\tau} \Phi\left(\frac{x}{\sqrt{2\tau}} + \beta\sqrt{2\tau}\right). \end{aligned}$$

The second integral is identical, except that β is replaced with α. Hence

$$I_2 = e^{\alpha x + \alpha^2\tau} \Phi\left(\frac{x}{\sqrt{2\tau}} + \alpha\sqrt{2\tau}\right).$$

Recall that $x = \ln\frac{S}{K}$, $k = \frac{2r}{\sigma^2}$, $\alpha = \frac{1}{2}(k-1) = \frac{r - \sigma^2/2}{\sigma^2}$, $\beta = \frac{1}{2}(k+1) = \frac{r + \sigma^2/2}{\sigma^2}$, and $\tau = \frac{1}{2}\sigma^2(T-t)$. Consequently, we have that

$$\frac{x}{\sqrt{2\tau}} + \beta\sqrt{2\tau} = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1$$

and that

$$\frac{x}{\sqrt{2\tau}} + \alpha\sqrt{2\tau} = d_1 - \sigma\sqrt{T-t} = d_2.$$

Hence the first integral is

$$I_1 = \exp\left(\beta x + \beta^2\tau\right)\Phi\left(d_1\right).$$

The second integral is identical except that β is replaced by α and involves d_2 instead of d_1

$$I_2 = \exp\left(\alpha x + \alpha^2\tau\right)\Phi\left(d_2\right).$$

The solution is therefore

$$\begin{aligned} W(x,\tau) &= I_1 - I_2 \\ &= e^{\beta x + \beta^2\tau}\Phi\left(d_1\right) - e^{\alpha x + \alpha^2\tau}\Phi\left(d_2\right). \end{aligned} \quad (46)$$

The solution in Equation (46), expressed in terms of I_1 and I_2, is the solution for $W(x,\tau)$. To obtain the solution for the call price $V(S_t,t)$ we must use Equation (44) and transform the solution in (46) back to V. From (44) and (46)

$$\begin{aligned} V(S,t) &= Ke^{-\alpha x - \beta^2\tau}W(x,\tau) \\ &= Ke^{-\alpha x - \beta^2\tau}\left[I_1 - I_2\right]. \end{aligned} \quad (47)$$

The first integral in Equation (47) is

$$\begin{aligned} Ke^{-\alpha x - \beta^2\tau}e^{\beta x + \beta^2\tau}\Phi\left(d_1\right) &= Ke^{(\beta-\alpha)x}\Phi\left(d_1\right) \\ &= S\Phi\left(d_1\right) \end{aligned} \quad (48)$$

since $\beta - \alpha = 1$. The second integral in Equation (47) is

$$\begin{aligned} Ke^{-\alpha x - \beta^2\tau}e^{\alpha x + \alpha^2\tau}\Phi\left(d_2\right) &= Ke^{\left(\alpha^2 - \beta^2\right)\tau}\Phi\left(d_2\right) \\ &= Ke^{-r(T-t)}\Phi\left(d_2\right) \end{aligned} \quad (49)$$

since $\alpha^2 - \beta^2 = -\frac{2r}{\sigma^2}$. Combining the terms in Equations (48) and (49) produces the Black-Scholes call price in Equation (1).

8 Black-Scholes From CAPM

8.1 The CAPM

The Capital Asset Pricing Model (CAPM) stipulates that the expected return of a security i in excess of the risk-free rate is

$$E\left[r_i\right] - r = \beta_i\left(E\left[r_M\right] - r\right)$$

where r_i is the return on the asset, r is the risk-free rate, r_M is the return on the market, and

$$\beta_i = \frac{Cov\left[r_i, r_M\right]}{Var\left[r_M\right]}$$

is the security's beta.

8.2 The CAPM for the Assets

In the time increment dt the expected stock price return, $E[r_S dt]$ is $E\left[\frac{dS_t}{S_t}\right]$, where S_t follows the diffusion in Equation (2). The expected return is therefore

$$E\left[\frac{dS_t}{S_t}\right] = rdt + \beta_S \left(E\left[r_M\right] - r\right) dt. \quad (50)$$

Similarly, the expected return on the derivative, $E[r_V dt]$ is $E\left[\frac{dV_t}{V_t}\right]$, where V_t follows the diffusion in (3), is

$$E\left[\frac{dV_t}{V_t}\right] = rdt + \beta_V \left(E\left[r_M\right] - r\right) dt. \quad (51)$$

8.3 The Black-Scholes PDE from the CAPM

Divide by V_t on both sides of the second line of Equation (3) to obtain

$$\frac{dV_t}{V_t} = \frac{1}{V_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\frac{dS_t}{S_t}\frac{S_t}{V_t},$$

which is

$$r_V dt = \frac{1}{V_t}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \frac{\partial V}{\partial S}\frac{S_t}{V_t} r_S dt. \quad (52)$$

Drop dt from both sides and take the covariance of r_V and r_M, noting that only the second term on the right-hand side of Equation (52) is stochastic

$$Cov\left[r_V, r_M\right] = \frac{\partial V}{\partial S}\frac{S_t}{V_t} Cov\left[r_S, r_M\right].$$

This implies the following relationship between the beta of the derivative, β_V, and the beta of the stock, β_S

$$\beta_V = \left(\frac{\partial V}{\partial S}\frac{S_t}{V_t}\right)\beta_S.$$

This is Equation (15) of Black and Scholes [1]. Multiply Equation (51) by V_t to obtain

$$\begin{aligned} E\left[dV_t\right] &= rV_t dt + V_t \beta_V \left(E\left[r_M\right] - r\right) dt \quad (53) \\ &= rV_t dt + \frac{\partial V}{\partial S} S_t \beta_S \left(E\left[r_M\right] - r\right) dt. \end{aligned}$$

This is Equation (18) of Black and Scholes [1]. Take expectations of the second line of Equation (3), and substitute for $E[dS_t]$ from Equation (50)

$$E\left[dV_t\right] = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S}\left[rS_t dt + S_t \beta_S \left(E\left[r_M - r\right]\right) dt\right] + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt. \quad (54)$$

Equate Equations (53) and (54), and drop dt from both sides. The term involving β_S cancels and we are left with

$$\frac{\partial V}{\partial t} + rS_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} - rV_t = 0. \tag{55}$$

We recognize that Equation (55) is the Black-Scholes PDE in Equation (33). Hence, we can obtain the Black-Scholes call price by appealing to the Feynman-Kac theorem exactly as was done in Section (6.2) and solving the integral as in Section (5).

9 Incorporating Dividends

The Black-Scholes call price in Equation (1) is for a call written on a non dividend-paying stock. There are two ways to incorporate dividends into the call price. The first is by assuming the stock pays a continuous dividend yield q. The second is by assuming the stock pays dividends in lump sumps, "lumpy" dividends.

9.1 Continuous Dividends

We assume that the dividend yield q is constant so that the holder of the stock receives an amount $qS_t dt$ of dividend in the time increment dt. After the dividend is paid out, the value of the stock drops by the dividend amount. In other words, without the dividend yield, the value of the stock increases by $rS_t dt$, but with the dividend yield the stock increases by $rS_t dt - qS_t dt = (r-q)\, S_t dt$. Hence, the expected return becomes $r - q$ instead of r, which implies that he risk-neutral process for S_t follows Equation (14) but with drift $r - q$ instead of r

$$dS_t = (r-q)\, S_t dt + \sigma S_t dW_t^{\mathbb{Q}}. \tag{56}$$

Following the same derivation in Section (3), Equation (56) has solution

$$S_T = S_t \exp\left(\left(r - q - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau^{\mathbb{Q}}\right)$$

where $\tau = T - t$. Hence, S_T follows the lognormal distribution with mean $S_t e^{(r-q)\tau}$ and variance $S_t^2 e^{2(r-q)\tau}\left(e^{\sigma^2\tau} - 1\right)$. Proceeding exactly as in Equation (18), the call price is

$$C\left(S_t, K, T\right) = e^{-r\tau} L_{S_T}(K) - e^{-r\tau}\left[1 - F(K)\right]. \tag{57}$$

The conditional expectation $L_{S_T}(K)$ from Equation (10) becomes

$$\begin{aligned} L_{S_T}(K) &= \exp\left(\ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)\tau + \frac{\sigma^2\tau}{2}\right) \\ &\quad \times\Phi\left(\frac{-\ln K + \ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) \\ &= S_t e^{(r-q)\tau}\Phi(d_1) \end{aligned} \tag{58}$$

with d_1 redefined as

$$d_1 = \frac{\ln\frac{S_t}{K} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Using Equation (7), the second term in Equation (18) becomes

$$\begin{aligned} e^{-r\tau}K\left[1 - F(K)\right] &= e^{-r\tau}K\left[1 - \Phi\left(\frac{\ln K - \ln S_t - \left(r - q - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)\right] \\ &= e^{-r\tau}K\Phi(d_2) \end{aligned} \tag{59}$$

with $d_2 = d_1 - \sigma\sqrt{\tau}$ as before. Substituting Equations (58) and (59) into Equation (57) produces the Black-Scholes price of a European call written on a stock that pays continuous dividends

$$C\left(S_t, K, T\right) = S_t e^{-q\tau}\Phi(d_1) - e^{-r\tau}K\Phi(d_2).$$

Hence, the only modification is that the current value of the stock price is decreased by $e^{-q\tau}$, and the return on the stock is decreased from r to $r-q$. All other computations are identical.

9.2 Lumpy Dividends

To come. Same idea: the current value of the stock price is decreased by the dividends, except not continuously.

References

[1] Black, F., and M. Scholes (1973). "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, Vol 81, No. 3, pp. 637-654.

[2] Hogg, R.V., and S. Klugman (1984). *Loss Distributions*. New York, NY: John Wiley & Sons.

[3] Hull, J. (2008). *Options, Futures, and Other Derivatives*, Seventh Edition. New York, NY: Prentice-Hall.

[4] Wilmott, P., Howison, S., and J. Dewynne (1995). *The Mathematics of Financial Derivatives: A Student Introduction*. Cambridge, UK: Cambridge University Press.